Black-Litterman model with intuitionistic fuzzy posterior return

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Abstract
The main objective of this paper is to present a variant of the Black-Litterman model. We consider the canonical expression of the model where the prior return is determined by means of the excess return from the CAPM market portfolio, derived using the reverse optimization method. As such, the prior return is at risk of quantified uncertainty. On the other hand, extensive discussion indicates that experts’ views are under Knightian uncertainty. For this reason, we propose a variant of the Black-Litterman model in which experts’ views are described as intuitionistic fuzzy numbers. The existence of posterior return is proved for this case and shown to be an intuitionistic fuzzy probabilistic set.

Keywords:
Black-Litterman model, Intuitionistic fuzzy number, Quantitative uncertainty, Knightian uncertainty, imprecision.

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El modelo Black-Litterman con retorno posterior intuicionista difuso

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Resumen
El principal objetivo de este artículo es presentar una cierta variante del modelo Black-Litterman. Se considera el caso canónico cuando el retorno a priori se determina por medio del diferencial de rentabilidad de la cartera de mercado CAPM que se obtiene utilizando el método de optimización revertida. Por tanto, el rendimiento a priori está bajo incertidumbre quantificable. Al mismo tiempo, un intenso debate muestra que las opiniones de los expertos están bajo incertidumbre knightiana. Por esta razón se propone esta variante del modelo Black-Litterman, en la cual las opiniones de los expertos se describen como un número intuicionista difuso. Se prueba la existencia, en este caso, de un retorno posterior y se muestra que es un conjunto probabilístico intuicionista difuso.

Palabras clave:
Modelo Black-Litterman, número difuso intuicionista, incertidumbre cuantitativa, incertidumbre knightiana, imprecisión.
1. Introduction

The Black-Litterman model (BLM hereafter) was introduced by Black and Litterman (1991a), expanded in Black and Litterman (1991b, 1992) and discussed in detail in Bevan and Winkelmann (1998), He and Litterman (2001), Litterman (2003) and Walters (2011). BLM combines the Capital Asset Pricing Model (CAPM) (Sharpe, 1964), reverse optimization (Sharpe, 1974), mixed estimation (Theil, 1971, 1978), the universal hedge ratio from Black’s global CAPM (Black, 1998a,b; Litterman, 2003), and mean-variance optimization (Markowitz, 1952). Many financial institutions use BLM for asset allocation as this model provides the flexibility of combining the market equilibrium with the investor’s views about the market.

There are many versions of BLM in the literature, and in each version the investor’s views are represented by a vector of random variables. This representation requires the assumption that the investor’s views are subject to quantitative uncertainty. However, this assumption is not empirically verifiable since investors’ views are very intuitive. Thus, we can only assume that investors’ views are subject to Knightian uncertainty (Knight, 1921).

Intuitionistic fuzzy sets (Atanassov and Stoeva, 1983) may be applied as an image of Knightian uncertainty. Therefore, the main aim of this article is to present the possibility of using intuitionistic fuzzy sets to describe investors’ views.

The rest of this article is organized as follows. After this introductory section, in Section 2 we describe the classical form of BLM. In Section 3 we present the basic concepts of intuitionistic fuzzy sets theory. The intuitionistic fuzzy investor views are discussed in Section 4. Section 5 provides final conclusions and recommendations for future research.

2. Black-Litterman model: the basic case

BLM uses the Bayesian approach to infer assets’ expected returns (Black and Litterman, 1991b). Under the Bayesian approach, the expected returns are themselves random variables; they are not observable and it is only possible to infer their probability distribution.

This inference starts with a prior belief and then additional information is incorporated in order to infer the posterior distribution. In BLM, the prior distribution is the CAPM equilibrium distribution and the investor’s views are the additional information.
The set \( \Omega \) is a set of all elementary states \( \omega \) of the financial market. Let us assume that there are \( n>1 \) assets in the market. The returns on these assets are represented by random variable \( \tilde{r}: \Omega \rightarrow \mathbb{R}^n \) which has a normal distribution with expected return \( \pi \) and covariance matrix \( \Sigma \). That is,

\[
\tilde{r} \sim N(\pi, \Sigma). \tag{1}
\]

The BLM uses ‘equilibrium’ returns as a neutral starting point. Equilibrium returns are the set of returns that clear the market. The equilibrium returns are derived using a reverse optimization method in which the vector \( \hat{\pi} \in \mathbb{R}^n \) of implied excess equilibrium returns is extracted from known information using formula

\[
\hat{\pi} = \lambda \cdot \Sigma \cdot w \tag{2}
\]

where \( \lambda \in \mathbb{R} \) is the risk aversion coefficient and \( w \in \mathbb{R}^n \) is the vector of market capitalization assets weights. The risk-aversion coefficient characterizes the expected risk-return tradeoff. It is the rate at which an investor will forego expected return for less variance. In the reverse optimization process, the risk aversion coefficient acts as a scaling factor for the reverse optimization estimate of excess returns; the weighted reverse optimized excess returns equal the specified market risk premium.

More often than not, investment managers have specific views regarding the expected return of some of the assets in a portfolio, which differ from the implied equilibrium return.

In addition to the CAPM prior, the investor also has \( k \geq 1 \) views on the market returns. Any view is expressed as a statement that for fixed \( i \leq k \) the linear combination of returns

\[
\tilde{v}_i = p_i^T \cdot \tilde{r} \tag{3}
\]

has a normal distribution with expected value \( \overline{v}_i \) and standard deviation \( \zeta_i \). The investor’s confidence in the view, \( \tilde{v}_i \), decreases as the standard deviation \( \zeta_i \) increases. Thus, the investor’s views can be expressed as system of linear equations:

\[
P \cdot \tilde{r} = \tilde{v}, \tag{4}
\]

where

\[
\tilde{v} \sim N(\overline{v}, \Xi) \tag{5}
\]
and
\[ P = [p_1^T, p_2^T, \ldots, p_k^T], \quad \bar{v} = (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k)^T, \]
\[ \mathbf{w} = (\omega_1, \omega_2, \ldots, \omega_k)^T, \quad \Xi = \begin{bmatrix} \zeta_1 & 0 & \cdots & 0 \\ 0 & \zeta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta_k \end{bmatrix} \]
(6)

Taking into account a priori returns in addition to the investor’s views, we can obtain a posterior return that has a normal distribution with expected return \( \pi_{BL} \) and covariance matrix \( \Sigma_{BL} \). That is
\[ \mathbf{f}_{BL} \sim N(\pi_{BL}, \Sigma_{BL}), \]
(7)
where
\[ \Sigma_{BL} = ((\tau \cdot \Sigma)^{-1} + P^T \cdot \Xi^{-1} \cdot P)^{-1} \]
(8)
\[ \pi_{BL} = \Sigma_{BL} \cdot ((\tau \cdot \Sigma)^{-1} \cdot \pi + P^T \cdot \Xi^{-1} \cdot \mathbf{w}) \]
(9)
for fixed scalar \( \tau \in \mathbb{R}^+ \). Walters (2011) states that the meaning and impact of the parameter \( \tau \) causes a great deal of confusion for many users of the BLM. Nevertheless, it can be said that the confidence in the prior expected return \( \pi \) versus investor’s views decreases as parameter \( \tau \) increases.

The two parameters of the BLM that control the relative importance placed on the equilibrium returns versus the investor’s views, the scalar \( \tau \) and the covariance matrix \( \Xi \), are very difficult to specify. Litterman and the Quantitative Resources Group, Goldman Sachs Asset Management (2003), point out that, “how to specify standard deviations \( \zeta_i \)” is a common question without a “universal answer”. Regarding \( \Xi \), Herold (2003) argues that the major difficulty of BLM is that it forces the user to specify a probability density function for each view, which makes BLM suitable only for quantitative managers.

### 3. Intuitionistic fuzzy sets in the real line: basic concepts

Let us consider the space of all real numbers \( \mathbb{R} \). The basic tool for imprecise classification of real numbers is the concept of fuzzy set \( A \subseteq \mathbb{R} \) which may be described as the set of ordered pairs
\[ A = \{(x, \mu_A(x)) : x \in \mathbb{R}\}, \]
(10)
where $\mu_A : \mathbb{R} \rightarrow [0, 1]$ is its membership function. An intuitionistic fuzzy set (IFS for short) $A \subseteq \mathbb{R}$ is defined as the set of ordered triples

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in \mathbb{R}\}, \quad (11)$$

where the non-membership function $\nu_A : \mathbb{R} \rightarrow [0, 1]$ fulfils the condition

$$\nu_A(x) \leq 1 - \mu_A(x) \quad (12)$$

for each $x \in \mathbb{R}$. The family of all IFS in the real line $\mathbb{R}$ is denoted by the symbol $\mathcal{I}(\mathbb{R})$ (Atanassov and Stoeva, 1983).

We define the hesitation function $\pi_A : \mathbb{R} \rightarrow [0, 1]$ by the identity

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x). \quad (13)$$

The value $\pi_A(x)$ indicates the degree of hesitation in assessing the relationship between the real number $x \in \mathbb{R}$ and IFS $A$. The hesitation function $\pi_A$ may thus be interpreted as a image of Knightian uncertainty (Knight, 1921).

For any $A, B \in \mathcal{I}(\mathbb{R})$ set theory operations are defined in the following way:

$$A^C = \{(x, \nu_A(x), \mu_A(x)) : x \in \mathbb{R}\}, \quad (14)$$

$$A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)) : x \in \mathbb{R}\}, \quad (15)$$

$$A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)) : x \in \mathbb{R}\}. \quad (16)$$

Let us consider the fuzzy subset $B$ described by its membership function $\mu_B : \mathbb{R} \rightarrow [0, 1]$. This fuzzy subset can be identified with IFS represented by the set of ordered triples

$$B^* = \{(x, \mu_B(x), 1 - \mu_B(x)) : x \in \mathbb{R}\}, \quad (17)$$

The hesitation function of the above IFS identically fulfils the condition

$$\pi_B(x) = 0. \quad (18)$$

This implies that application of fuzzy sets for modelling real-life situations entails the strong assumption that we can always decide on the fulfilment of the required conditions by each elementary state. As we know from everyday observations, however, this is not usually the case, and our decisions carry a noticeable hesitation margin.
This means that the extension of the fuzzy sets class to the IFS class extends the capabilities of the model for generating a reliable description of imprecision.

IFSs are used to describe imprecise information under Knightian uncertainty. Many papers on this subject (e.g. Klir, 1993) distinguish between two components of imprecision: ambiguity and indistinctness. Ambiguity in information is interpreted as a lack of clear indication in support of one option over the various alternatives available, whereas we interpret indistinctness as the lack of an explicit distinction between the given information and its negation. The hesitation function describes information undecidability, which is interpreted as the impossibility of deciding whether each elementary state fulfils the postulated requirements, thus causing Knightian uncertainty.

If information imprecision or undecidability increases, it reduces the usefulness of the information, giving rise to problems with the evaluation of these phenomena.

In this paper, we use the following measures suggested in Piasecki (2015):

Ambiguity is evaluated by energy measure \( d : \mathcal{I}(\mathbb{R}) \rightarrow [0; 1] \) given by the identity

\[
d(A) = \lim_{\gamma \to +\infty} \frac{\int_{-\gamma}^{\gamma} \mu_A(x) dx}{1 + \int_{-\gamma}^{\gamma} \mu_A(x) dx}.
\]  

(19)

Indistinctness is measured by the most popular entropy \( e : \mathcal{I}(\mathbb{R}) \rightarrow [0; 1] \) which is defined by Kosko (1986) as follows:

\[
e(A) = \frac{d(A \cap A^c)}{d(A \cup A^c)}.
\]  

(20)

Undecidability is evaluated by the ignorance measure \( k : \mathcal{I}(\mathbb{R}) \rightarrow [0; 1] \) given by the identity

\[
k(A) = d(flA) - d(yA)
\]  

(21)

where according to Atanassov (1993), for any IFS \( A \in \mathcal{I}(\mathbb{R}) \) we have:

\[
flA = \{(x, \mu_A(x), 1 - \mu_A(x)) : x \in \mathbb{R}\},
\]  

(22)

\[
yA = \{(x, 1 - \upsilon_A(x), \upsilon_A(x)) : x \in \mathbb{R}\}.
\]  

(23)

An increase in imprecision or in undecidability significantly worsens the information quality. Thus, using the vector-valued function \((d(\cdot), e(\cdot), k(\cdot))\) facilitates information quality management. Here it is desirable to minimize the value of each coordinate.
4. Intuitionistic fuzzy posterior return

Let us turn once again to investor views, which are additional information in BLM. In Section 2, each investor’s view is represented by a random variable. Obviously, the probability distribution of any investor’s view is unobservable; thus we can say that each investor’s view is subject to Knightian uncertainty. This implies that:

- Investors’ views cannot be represented by a random variable.
- Any investor’s view may be represented by IFS in the real line.

Therefore, in (3) the random variable $\tilde{v}_i$ should be replaced by the IFS $V_i \in \mathcal{I}(\mathbb{R})$. Then we obtain the following condition:

$$V_i = p_i^T \cdot \tilde{r}(\omega), \quad (24)$$

where $\tilde{r}(\omega) = (\tilde{r}_1(\omega), \tilde{r}_2(\omega), \ldots, \tilde{r}_n(\omega))^T$ is a return determined for the fixed elementary state $\omega \in \Omega$. Obviously, coordinate $\tilde{r}_i(\omega)$ is not a real number, thus the vector $\tilde{r}(\omega)$ is not a realization of a random variable. The IFS $V_i$ membership function describes the distribution of possible values of investor views. The IFS $V_i$ non-membership function describes the distribution of impossible values of investor views. The importance of each investor’s view depends on the usefulness of this view. Thus, the importance of the investor’s view is evaluated by means of the vector $(d(V_i), e(V_i), k(V_i))$. The importance of the view decreases with any increase in any coordinate of this vector. The investor’s view may not be an intuitionistic fuzzy number (Burillo et al., 1994).

For example, the IFS $V_i$ may be given as expected return rates dependent on expected future value and intuitionistic fuzzy present value (Piasecki, 2015). Moreover, intuitionistic fuzzy present value can be determined as behavioural present value (Piasecki, 2013), which explicitly depends on observed market price and on the impact of market conditions on the investor’s beliefs. All this proves that IFS $V_i$ can be strictly determined as a value which is verifiable.

For a fixed elementary state $\omega \in \Omega$, immediately from (24) we obtain the system of linear equations:

$$P \cdot \tilde{r}(\omega) = V, \quad (25)$$

where $V = (V_1, V_2, \ldots, V_k)^T \in [\mathcal{I}(\mathbb{R})]^k$. Pradham and Pal (2014) show that the system of equations (25) is a solvable system. Solution uniqueness is not discussed in Pradham and Pal (2014); therefore, let us consider the general solution as the indexed family of particular solutions:
\[ \mathbf{R}^{(\lambda)}(\omega) = (R_1^{(\lambda)}(\omega), R_2^{(\lambda)}(\omega), \ldots, R_n^{(\lambda)}(\omega))^T \in [\mathcal{P}(\mathbb{R})]^n, \]

where \( \lambda \in \Lambda \). Each IFS \( R_i^{(\lambda)}(\omega) \) is represented by its conditional membership function \( \rho_i^{(\lambda)}(\cdot|\omega) \colon \mathbb{R} \rightarrow [0, 1] \) and its conditional non-membership function \( q_i^{(\lambda)}(\cdot|\omega) \colon \mathbb{R} \rightarrow [0, 1] \).

Let us now consider the indexed family

\[ \tilde{R}_i^{(\lambda)} = \{ R_i^{(\lambda)}(\omega) : \omega \in \Omega \}. \]

which is one alternative of posterior return on assets indexed by \( i < n \). This posterior return is the intuitionistic fuzzy probabilistic set (Zhang et al. 2009) represented by its membership function \( \rho_i^{(\lambda)} : \mathbb{R} \times \Omega \rightarrow [0, 1] \) determined by the identity

\[ \rho_i^{(\lambda)}(x, \omega) = \rho_i^{(\lambda)}(x|\omega) \]

and by its non-membership function \( q_i^{(\lambda)} : \mathbb{R} \times \Omega \rightarrow [0, 1] \) determined by the identity

\[ q_i^{(\lambda)}(x, \omega) = q_i^{(\lambda)}(x|\omega). \]

Let posterior return on assets indexed by \( i \leq n \) be denoted by the symbol \( \tilde{R}_i \). The posterior return \( \tilde{R}_i \) is equal to union of all its alternatives \( \tilde{R}_i^{(\lambda)} \). Thus, its posterior return is represented by its membership function \( \rho_i : \mathbb{R} \times \Omega \rightarrow [0, 1] \) determined by the identity

\[ \rho_i(x, \omega) = \sup \{ \rho_i^{(\lambda)}(x, \omega) : \lambda \in \Lambda \} \]

and by its non-membership function \( q_i : \mathbb{R} \times \Omega \rightarrow [0, 1] \) determined by the identity

\[ q_i(x, \omega) = \inf \{ q_i^{(\lambda)}(x, \omega) : \lambda \in \Lambda \}. \]

Finally, we obtain the posterior return given as vector

\[ \tilde{\mathbf{R}} = (\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n)^T. \]

Immediately from (25) we determine that the probability measure \( \mathcal{P} : 2^\Omega \supset \sigma \rightarrow [0, 1] \) is uniquely defined by the prior distribution that is CAPM equilibrium distribution.

In this way, we gathered all the information necessary for the analysis of the intuitionistic fuzzy return rate as described in Piasecki (2016).
5. Conclusions

In this paper, BLM is modified such that randomized investor views are replaced by intuitionistic fuzzy views. This replacement is justified by the observation that investors’ views are subject to Knightian uncertainty. In this way, we obtain the model independent of the two parameters which control the relative importance placed on the equilibrium returns versus the investor’s views, the scalar $\tau$ and the covariance matrix $\mathbf{\Sigma}$. Let us recall that the meaning and impact of the first parameter causes a great deal of confusion for many BLM users (Walters, 2011). Moreover, in the subject literature we cannot find a well-justified method of covariance matrix $\mathbf{\Sigma}$ estimation. Recapitulating, eliminating these parameters allows us to replace the BLM with the modified BLM, which is free from subjective evaluations of the significance of investor views.

This represents the basic advantage of the proposed modified BLM. In line with the suggestion made in Markowitz (2012), the BLM with intuitionistic fuzzy investor views presented here may be generalized for use in cases when the a priori return has a non-Gaussian probability distribution.

In this article, we have proved only that posterior return exists. Therefore, the results obtained may be used in finance theory only as a prescriptive model. On the other hand, these results can be directly used in the decision-making models described in Li (2014), meaning that the findings presented above can constitute a theoretical foundation for constructing an investment decision support system.

Applications of the normative model presented above involve several difficulties. The main difficulty is the high formal and computational complexity of the tasks of determining the membership and non-membership functions of the posterior return. Computational complexity of the normative model is the price we pay for the lack of detailed assumptions about investor views. On the other hand, low logical complexity is an important advantage of the formal model presented in this paper.

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