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# Testing an Innovative Variance Reduction Technique for Pricing Bond Options in the Framework of the CIR Model

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## Abstract

We design an innovative variance reduction technique coupled with Monte Carlo (MC) simulation that prices accurately plain-vanilla zero coupon bond options. This technique speeds up the convergence of the simulation and offers better results than MC simulation using antithetic variables. Our benchmark is the closed-form solution of Cox Ingersoll and Ross (CIR, 1985). Our paper shows that, when pricing bond options with MC simulation, we can constrain the Wiener process inside upper and lower bands to speed up the convergence towards the 'true' option value (the CIR analytical solution). Furthermore, it works best when the bands are drawn at plus or minus 0.5 standard deviations and our technique is less time consuming than a plain MC simulation. Finally, we introduce an original stochastic fifth-order polynomial model beside the CIR solution. Our contribution is to provide market practitioners with an efficient variance reduction technique, easy to implement. The major challenge of our technique would be to price bond options in times of high market volatility, when option price needs badly to reflect rare events located in the tails of the distribution. However, we can argue that pricing options in times of volatile markets is a challenge for every option pricing model.

## Keywords:

Variance reduction technique; Importance sampling; Cox-Ingersoll-Ross model; Polynomial model; Bond option; Monte Carlo simulation.

***JEL classification:*** C15; C63; G13

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# Una nueva técnica de reducción de la varianza para **valorar opciones sobre bonos en el marco del modelo CIR**

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**Resumen**

En este artículo se diseña una nueva técnica de reducción de la varianza combinada con simulación de Monte Carlo que valora con exactitud las opciones sobre bonos cupón cero “plain vanilla”. Esta técnica acelera la convergencia de la simulación y ofrece mejores resultados que la simulación Monte Carlo con variables antitéticas. El objetivo que se persigue es la solución en forma cerrada de Cox, Ingersoll y Ross (CIR, 1985). Se muestra que cuando se valoran opciones sobre bonos con simulación Monte Carlo, se puede limitar el proceso de Wiener dentro de las bandas superior e inferior para acelerar la convergencia hacia el 'verdadero' valor de la opción (la solución analítica CIR). Además, cuando mejor funciona esta alternativa es cuando las bandas se sitúan a más o menos 0,5 desviaciones típicas y es computacionalmente más rápida que la simulación Monte Carlo normal. Finalmente, se introduce un modelo estocástico original, polinomial de quinto orden, junto con la solución CIR. La contribución de este artículo es proporcionar a los profesionales del mercado una técnica de reducción de la variabilidad eficiente y fácil de implementar. El principal reto de esta técnica es la valoración de opciones sobre bonos en periodos de alta volatilidad, cuando la valoración de opciones necesita imperiosamente reflejar acontecimientos raros localizados en las colas de la distribución. Sin embargo, se puede argüir que la valoración de opciones cuando los mercados tienen elevada volatilidad es un reto para cada modelo de valoración de opciones.

**Palabras clave:**

Técnica de reducción de la varianza, muestreo por importancia; modelo Cox-Ingersoll-Ross, modelo polinomial, opción sobre bono, simulación Monte Carlo.

## 1. Introduction

Monte Carlo (MC) simulation is widely used in the industry for easiness of implementation, especially for pricing exotic options (barrier, lookback, digital, Asian, basket, etc). The closed-form solution is not always available for this type of options. In this situation, MC simulation becomes helpful. In this paper, our main contribution is to propose a numerical solution to price bond options in the frameworks of Cox Ingersoll and Ross model (CIR, 1985). We apply an innovative variance reduction technique, easily implemented, that is not time-consuming during the MC simulation and that, as expected, speeds up the convergence towards the 'true' solution.

Market practitioners may find variance reduction techniques difficult to implement. Besides antithetic variable and our innovative technique, other variance reduction techniques may present implementation problems. For example, the use of control variate techniques in option pricing presents the following problems: the analytical solution of the Delta, Gamma or Vega of the option may not be readily available. Moreover, if the practitioner chooses a numerical approximation of the option Greeks, it may not be possible to implement it with the control variate technique. Using the CIR model example, we obtain a yield curve by simulating the stochastic equation. The simulated curve provides interest rate values that are inputs of the bond price computation, which is, in turn, an input of the option price computation. However, typical numerical Greeks approximation that helps computing a control variate is obtained step by step during the simulation of a given trajectory and the control variate is a direct input of the option price computation. Therefore, this control variate method is not applicable to the CIR model when pricing a bond option with MC simulation. We compare our innovative numerical solution to the solution of an original stochastic fifth-order polynomial model presented in this paper. The reason to propose a challenger to the CIR model is to bring perspectives to the results with a distinct model, which does not include the mean reverting process common to CIR (1985), Vasicek (1977), Ho and Lee (1986), Hull and White (1990), Fong and Vasicek (1992), etc., models.

Our study is organised as follows. The literature review highlights our choice of the polynomial model as an alternative model to the CIR model. This section also reviews the *Importance Sampling* procedure on which we base our variance reduction technique. The methodology section presents our models. We wrap up our results and we make final comments in the two last sections.

## 2. Literature review

### 2.1 Polynomial models

We compare our variance reduction technique coupled with MC simulation and the CIR model (1985) to an original stochastic fifth-order polynomial model also based on MC simulation. Several authors have proposed polynomial models to fit the yield curve. At the start of the polynomial models incubation period, pioneers like Kornbluth and Salkin (1992) discussed the effect of various polynomial representations of the yield curve on the tilting of medium to long term bond portfolios. Pham (1998) proposed 'a methodology of fitting the term structure of interest rates with Chebyshev polynomials incorporated into a quantity called the interest cumulator and then subjected to a minimization procedure to yield parameters that subsequently maps out zero-coupon yield curves'. Almeida *et al.*, (1998), proposed modelling the term structure of interest rates R(·) as a linear combination of Legendre polynomials. Bing-Huei (1999) used 'curve fitting techniques with the observed government coupon bond prices to estimate the term structure. Bing-Huei applied the B-spline functions to approximate the discount function, spot yield curve, and forward yield curve respectively'.

Bali and Wu (2006) provided a comprehensive analysis of the short-term interest-rate dynamics. They think of 'the drift function as a Laurent series expansion of a generic function with positive order of five and negative order of one'. They tested a fifth-order polynomial function of the interest rate. They used the fifth-order polynomial drift specification as a general nonlinear specification $\mu(r_t)=\alpha_0+\alpha_1 r_t+\alpha_2 r_t^2+\alpha_3 r_t^3+\alpha_4 r_t^4+\alpha_5 r_t^5$ and contrasted it with the affine[1] specification $\mu(r_t)=\alpha_0+\alpha_1 r_t$. They concluded that 'nonlinearity exists in the very short-term interest rates process due to different speeds of mean reversion at different interest-rate levels. This difference becomes smaller for longer-maturity interest rates due to the smoothing effect of market expectations. In conclusion, it is more difficult to identify nonlinearities in the longer-term interest rates than in the very short-term ones'.

In line with Bali and Wu (2006), we choose to fit the yield curve with a fifth-order polynomial function but we add an innovation term to the function as it appears in equation 4 below in order to simulate the yield curves.

### 2.2 Importance Sampling

Since our variance reduction technique focuses on the innovation term of the CIR model and can be classified as *'Importance Sampling'*, we briefly review the topic discussed in the literature. Pioneers such as Glynn and Iglehart (1989) promoted the

[1] *For a discussion on affine term structure models see Jakas (2011).*

*Importance Sampling* procedure as a variance reduction technique for increasing the efficiency of MC simulation. Chang *et al.* (2006) showed that '*Importance Sampling*, as an efficiency improvement technique for MC simulations, is particularly well-suited for correlation products whose payoffs are contingent on the occurrence of rare events'. One must choose the *Importance Sampling* distribution carefully in order to achieve variance reductions. In other respects, Capriotti (2008) described 'a simple *Importance Sampling* strategy for MC simulations based on a least-squares optimization procedure'. Least-squares Importance Sampling (LSIS) is especially efficient when we need 'to adjust higher moments of the sampling distribution, or to deal with non-Gaussian or multi-modal densities, in order to achieve variance reductions'.

Gouda and Szántai (2008) focused on the manipulation of the innovation term by dealing with estimation of rare event probabilities in stochastic networks. Their 'main idea is to simulate the random system under a modified set of parameters, so as to make the occurrence of the rare event more likely'. In our paper, we just do the opposite by removing the rare events from the distribution. Neddermeyer (2009) stated that 'the variance reduction established by the *Importance Sampling* procedure strongly depends on the choice of the *Importance Sampling* distribution. A good choice is often hard to achieve especially for high-dimensional integration problems'. This is why Neddermeyer proposes a nonparametric estimation of the optimal Importance Sampling distribution called "nonparametric Importance Sampling", as a substitute to parametric approaches. Regarding the nature of the distribution of the innovation term, Hou and Suardi (2011) used the Student's t distribution for interest rate innovation and argue that it is consistent with the widely observed non-normal short-term interest rate distribution. In our paper, we rely on the assumption of normality of the innovation term.

**2.3. Adding bands to the simulation**

The intuition of adding lower and upper bounds to interest rates was proposed by Delbaen and Shirakawa (2002) with a new interest rate dynamics model where the interest rate fluctuates in a bounded region. The equation of the short-term interest rate becomes:

$$dr = \alpha(r_\mu - r_t)dt + \beta\sqrt{(r_t - r_m)(r_M - r_t)}.\ dW_t$$

with $r_m < r_\mu < r_M$. Théoret and Rostan (2005) proposed a variance reduction technique that constrains the short-term interest-rate inside Bollinger bands during a MC simulation. Our innovative variance reduction technique is based on a bounded distribution of the innovation term $\varepsilon$. During the MC, we draw $\varepsilon$ from a Normal distribution $N(0,1)$. To speed up the convergence of the simulation, we simply reduce the interval of drawing, for instance $\varepsilon \in [-1,1]$. This is *Importance Sampling* since we constraint the distribution inside upper and lower bands.

## 3. Methodology

To assess the ability of our innovative technique to reduce the variance of the simulated paths distribution, we price a bond option with the closed-form solution of Cox, Ingersoll and Ross model (CIR, 1985) called the 'true' solution. Then, we price the same option with MC simulation using: 1) CIR and 2) An original stochastic fifth-order polynomial model, with and without our variance reduction technique.

Cox, Ingersoll and Ross, (1985), propose to capture the behaviour of the short-term interest rate with the following process:

$$dr = \alpha(\mu - r)dt + \sigma\sqrt{r}\, dz_1 \qquad (1)$$

with $dz_1 = \varepsilon\sqrt{dt}$ .

They present a closed-form solution for a European call option written on a pure discount bond. We use this solution as the 'true' solution to show that our variance reduction technique coupled with MC simulation accurately prices bond options, reducing significantly the simulation time.

Our methodology follows three steps:

### 3.1 Step 1

We simulate a yield curve with $\alpha = 1$, $\mu = 0.04$, $\sigma = 0.015$ and $r_0 = 0.02$, $r_0$ being the initial interest rate at time $T = 0$. We obtain the yield curve (Figure 1). This yield curve is our reference curve, from which we compute the value of a European call option on a pure discount bond with analytical and numerical solutions.

Figure 1. **Simulation of a yiel curve over 5 years when $\alpha = 1$, $\mu = 0.04$ and $\sigma = 0.015$ and $r_0 = 0.01$, $dt = 0.01$ for 500 steps**


SOURCE: ROSTAN AND ROSTAN

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### 3.2 Step 2

We calibrate equation 1 in order to fit the yield curve presented in Figure 1, i.e. to find parameters $\alpha$, $\mu$ and $\sigma$ corresponding to the maximization of the log-likelihood function (equation 2). We apply Kladivko's (2007) methodology. The log-likelihood function of the CIR process is:

$$lnL(\theta)=(N-1)\ ln\ c+\sum_{i=1}^{N-1}\left\{u_{t_i}+v_{t_i+1}+0.5q ln\left(\frac{v_{t_i+1}}{u_{t_i}}\right)+ln\{I_q\left(2\sqrt{u_{t_i}v_{t_i+1}}\right)\}\right\} \quad (2)$$

where $u_{t_i}=cr_{t_i}e^{-\alpha\Delta t}$ and $v_{t_i+1}=cr_{t_i+1}$. We find maximum likelihood estimates $\hat{\theta}$ of parameter vector $\theta$ by maximizing the log-likelihood function (2) over its parameter space:

$$\theta \equiv (\alpha, \mu, \sigma)=\arg\max_{\theta}\ ln\ L(\theta) \quad (3)$$

Since the logarithmic function is monotonically increasing, maximizing the log-likelihood function also maximizes the likelihood function. Refer to Kladivko's (2007) methodology, for the practical implementation of the calibration. Three parameters values result from the optimization of the maximum likelihood objective function: $\alpha = 0.9204$, $\mu = 0.0392$, $\sigma = 0.0147$.

In our paper, we also feature an original stochastic fifth-order polynomial model to capture the yield curve[2]. Equation 4 illustrates the model:

$$r_t=p_1t^5+p_2t^4+p_3t^3+p_4t^2+p_5t^1+p_6+\sqrt{\tilde{r}_t}.\tilde{\sigma}.\varepsilon \quad (4)$$

with $p_1, p_2, p_3, p_4, p_5, p_6$ the fifth-order polynomial coefficients that fit the observed yield curve in a least-squares sense; $\tilde{r}_t$ is the fifth-order polynomial interest rate estimate at time $t$. Equation 4 models the trend –the drift function– with the fifth-order polynomial fit of the yield curve. The square root diffusion process (the last term of the equation) is borrowed from the CIR model. We do not take into account the step $\sqrt{dt}$ in the diffusion process, since equation 4 models $r_t$ and not $dr_t$.

The annualized volatility of the short-term interest-rate ($\tilde{\sigma}$) is obtained using equation 5:

$$\tilde{\sigma} = \sqrt{\frac{1}{N}\sum_{t=1}^{N}(r_t-\tilde{r}_t)^2}\ .\sqrt{252} \quad (5)$$

With $\tilde{r}_t$ the fifth-order polynomial interest rate estimate at time $t$ and $r_t$ the rate at time $t$ of the observed yield curve. Fitting the simulated yield curve, we obtain $p_1 = 4.9368e\text{-}005$, $p_2 = -0.0004$, $p_3 = 0.0011$, $p_4 = -0.0024$, $p_5 = 0.0115$, $p_6 = 0.0207$ and sigma = 0.0116.

[2] *See Jakas (2011) for an alternative approach to term structure modelling.*

### 3.3. Step 3

We price a two-year European call option with a strike of 0.67 on a five-year pure discount bond with a face value of 1 $ (e.g. Clewlow and Strickland, 1998). Using the CIR closed-form solution with the following inputs obtained from the maximum likelihood estimation of the Cox-Ingersoll-Ross process: $\alpha$ = 0.9204, $\mu$ = 0.0392, $\sigma$ = 0.0147, and $r$ = 0.02 the instantaneous short-rate, we get a European call option premium of 0.2088. We compare the analytical solution to the MC simulation with 500 steps and a time step $dt$ = 0.01 over 5 years, simulating Equation 1 and Equation 4, with and without our variance reduction technique.

Our variance reduction technique is based on a bounded distribution of $\varepsilon$. During the MC simulation, we draw ? from a Normal distribution $N(0,1)$. To speed up the convergence of the simulation, we simply reduce the interval of drawing, for instance $\varepsilon \in [-1,1]$. To implement this variance reduction in Matlab, we write the following line, for example with $\varepsilon \in [-0.5,+0.5]$:

*epsilon=norminv((normcdf(0.5)-(1- normcdf(0.5)))*rand+(1- normcdf(0.5)));*

Another example with $\varepsilon \in [-1,+1]$, the Matlab algorithm becomes:

*epsilon=norminv((normcdf(1)-(1- normcdf(1)))*rand+(1- normcdf(1)));*

## 4. Results

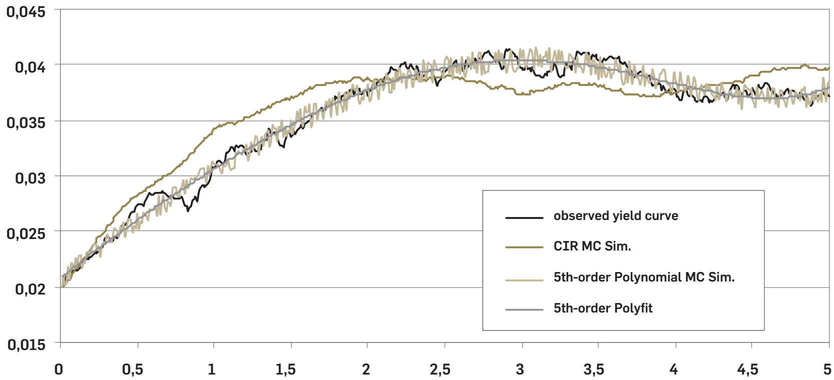
Given our initial example at step 3, we will need to compute the simulated 2-year and 5-year interest rates. The 2-year rate helps us discounting the expected value of the option payoff of a 2-year maturity. The 2- and 5-year spot rates, ${}_0r_2$ and ${}_0r_5$ respectively, help computing the forward rate ${}_2r_5$ (the spot rate of 3 years in 2 years) in order to compute the discount bond price in two years. Our computation is based on the following relationship:

$$(1+{}_0r_2)^2\,(1+{}_2r_5)^3 = (1+{}_0r_5)^5 \qquad (6)$$

Figure 2 illustrates two interest rate yield curves over a 5-year period obtained with Monte Carlo simulation using:

- Equation 1 with the 3 parameters obtained by optimization.
- Equation 4 with the fifth-order polynomial coefficients obtained by fitting the simulated yield curve and with the volatility sigma computed with Equation 5.

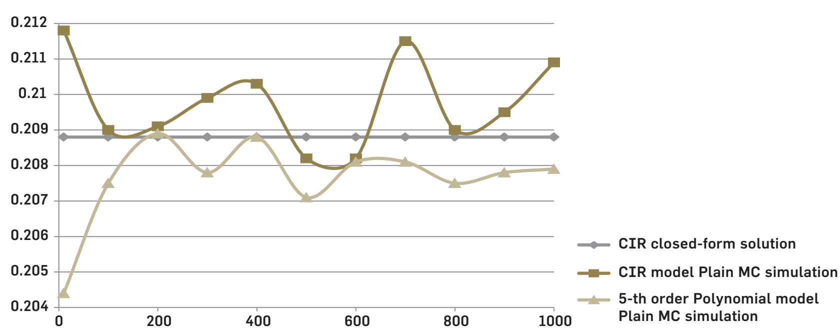
Figure 2. **Simulating two yield curves with MC simulation over 5 years: in brawn using equation 1 (CIR), in beige using equation 4 (Polynomial). In black, the initial simulated yield curve. $\varepsilon \in [-0.5,+0.5]$**




SOURCE: ROSTAN AND ROSTAN

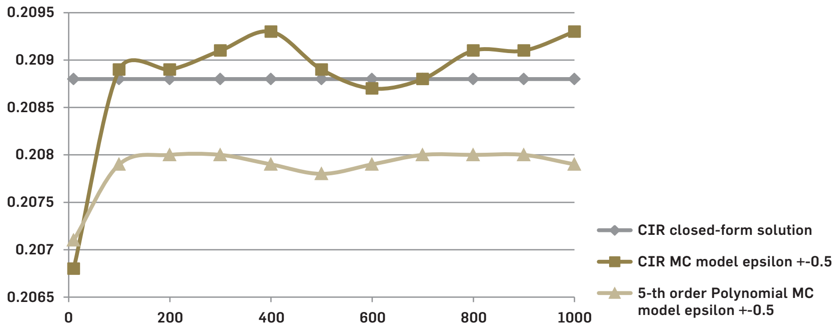
Figure 3 plots the convergence towards the closed-form solution value of plain MC simulations with no variance reduction technique, simulating equations 1 and 4, increasing the number of trajectories from 10 to 1,000 (X-axis), when the number of time steps is 500 ($dt = 0.01$). We observe that the fifth-order Polynomial model converges faster than the CIR model. After 600 simulations, the option values obtained with the fifth-order Polynomial model are stabilizing around 0.208 which is, as we see in Table 3 below, the best value that this model can display with a large number of simulations.

Figure 3. **Illustrating convergence of plain MC simulations –CIR and fifth-order Polynomial- with no variance reduction technique, towards the "true" value obtained with the CIR closed-form solution, when increasing the number of trajectories from 10 to 1,000.**




SOURCE: ROSTAN AND ROSTAN

**Figure 4. Illustrating convergence of MC simulations coupled with our innovative variance reduction technique with $\varepsilon$ [-0.5,0.5], towards the "true" value obtained with the CIR closed-form solution, when increasing the number of trajectories from 10 to 1,000.**




SOURCE: ROSTAN AND ROSTAN

Figure 4 illustrates the convergence towards the closed-form solution value of two MC simulations coupled with our variance reduction technique with $\varepsilon \in [-0.5,0.5]$, simulating equations 1 and 4, increasing the number of trajectories from 10 to 1,000 (X-axis), when the number of time steps is 500 ($dt = 0.01$). Our variance reduction technique significantly reduces the variance of the payoffs distribution, for both CIR and stochastic fifth-order Polynomial models, and therefore speeds up the convergence towards the true value. After 300 simulations, the option value stabilizes and is close to the "true" value. Furthermore, MC simulation using the CIR model is more accurate than using the Polynomial model. We can explain this fact intuitively since equation 1 is used for both the CIR closed-form solution and the MC simulation to simulate the short-term interest rate. Therefore, there is an evident convergence of analytical and numerical solutions that are based on the same CIR equation.

**Table 1. Options values obtained by increasing the number of simulations with MC methods, Plain MC, with our variance reduction technique, with antithetic variable. Results are compared to the CIR closed form solution. Number of time steps is 500 (dt = 0.01).**

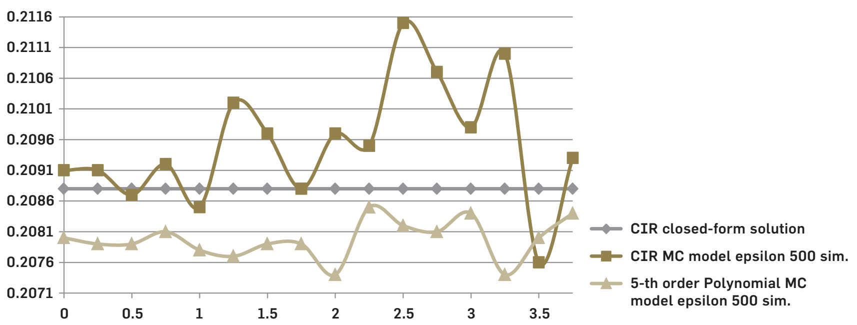
| Number of trajectories | CIR closed form solution | CIR model Plain MC simulation | Polynomial model Plain MC simulation | Simulation time in seconds | CIR model MC epsilon+ 0.5 | Polynomial model MC epsilon+ 0.5 | Simulation time in seconds | CIR model Antithetic MC simulation | Polynomial model Antithetic MC simulation | Simulation time in seconds |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0.2088 | 0.2118 | 0.2044 | 2 | 0.2068 | 0.2071 | 2 | 0.2097 | 0.208 | 3 |
| 100 | 0.2088 | 0.209 | 0.2075 | 12 | 0.2089 | 0.2079 | 13 | 0.2096 | 0.208 | 14 |
| 200 | 0.2088 | 0.2091 | 0.2089 | 24 | 0.2089 | 0.208 | 25 | 0.2095 | 0.208 | 27 |
| 300 | 0.2088 | 0.2099 | 0.2078 | 35 | 0.2091 | 0.208 | 36 | 0.2096 | 0.208 | 40 |
| 400 | 0.2088 | 0.2103 | 0.2088 | 46 | 0.2093 | 0.2079 | 48 | 0.2096 | 0.208 | 53 |
| 500 | 0.2088 | 0.2082 | 0.2071 | 57 | 0.2089 | 0.2078 | 59 | 0.2096 | 0.208 | 66 |
| 600 | 0.2088 | 0.2082 | 0.2081 | 68 | 0.2087 | 0.2079 | 71 | 0.2096 | 0.208 | 79 |
| 700 | 0.2088 | 0.2115 | 0.2081 | 79 | 0.2088 | 0.208 | 83 | 0.2096 | 0.208 | 93 |
| 800 | 0.2088 | 0.209 | 0.2075 | 91 | 0.2091 | 0.208 | 95 | 0.2096 | 0.208 | 104 |
| 900 | 0.2088 | 0.2095 | 0.2078 | 102 | 0.2091 | 0.208 | 107 | 0.2096 | 0.208 | 117 |
| 1000 | 0.2088 | 0.2109 | 0.2079 | 113 | 0.2093 | 0.2079 | 118 | 0.2096 | 0.208 | 133 |
| | RMSE: | 0.0051 | 0.0054 | RMSE: | 0.0022 | 0.0032 | RMSE: | 0.0027 | 0.0027 | |

Table 1 gathers the option values obtained with MC simulations with and without our variance reduction technique and the simulation time for an increasing number of simulations; results are compared to the CIR closed-form solution. In terms of simulation time, our variance reduction technique using $\varepsilon \in [-0.5,0.5]$ shows that it is as fast as the plain MC simulation. At this stage, two questions arise: 1) What is the optimal interval of epsilon for the solution to converge faster, ±1, ± 2? 2) How good is our reduction technique compared to an alternative reduction technique such as the antithetic variable technique?

### 4.1 Finding the optimal interval of epsilon

To find the optimal interval of epsilon in order to speed the convergence, we simulate 500 trajectories with the CIR and the stochastic fifth-order Polynomial models, both coupled with our variance reduction technique, making the interval of epsilon varying from 0 to ±4:

Figure 5. **Illustrating convergence of MC simulations coupled with our innovative variance reduction technique changing the interval of $\varepsilon$ varying from [0,0] to [-3.75, 3.75], X-axis, for 500 simulations. Number of time steps is 500 ($dt$ = 0.01).**




SOURCE: ROSTAN AND ROSTAN

We observe in Figure 5 that:

**1)** Both MC/CIR and MC/Polynomial models are more volatile when $\varepsilon \in [-2,2]$ and for wider intervals. Our variance reduction is working well with smaller intervals of epsilon since we observe less volatility in the results.

**2)** The fifth-order polynomial model always underestimates the 'true' value. The MC/CIR performs best for narrower intervals of epsilon.

**3)** When epsilon equals zero, i.e. $\varepsilon \in [0,0]$, the option value obtained with MC/CIR slightly overshoots the 'true value' at 0.2091. However, since the diffusion term in equations 1 and 4 equals zero, we do not talk anymore of MC simulation, since the trajectory is identical one simulation after another. Interestingly, results in Table

3 suggest that with a greater number of simulations (e.g. 5,000), MC/CIR values converge towards 0.2090. Thus, we can suggest that when pricing a plain vanilla zero coupon bond option, by taking only the drift function of equation 1 (the first part of the equation), by choosing epsilon equals zero, and by simulating one time, the process gives directly a value close to the best value that the Monte Carlo simulation can offer. This process is obviously sensitive to the calibration method that will greatly influence the result.

With epsilon equals to zero, $\varepsilon \in [0,0]$, we obtain the option values of Table 2 with the MC/Polynomial model, making the polynomial order varying from 2 to 8:

**Table 2. MC/Polymonial simulations when ee [0.0], i.e. when the diffusion parameter of equiation 4 is equal to zero; Number of simulations: 1. Number of time steps is 500 (*dt* = 0.01).**

| Polynomial Order: | CIR closed-form solution | MC/Polynomial option value |
|---|---|---|
| 2 | 0.2088 | 0.2193 |
| 3 | 0.2088 | 0.216 |
| 4 | 0.2088 | 0.2104 |
| 5 | 0.2088 | 0.208 |
| 6 | 0.2088 | 0.2106 |
| 7 | 0.2088 | 0.2102 |
| 8 | 0.2088 | 0.2122 |

SOURCE: ROSTAN AND ROSTAN

We observe that the *fifth*-order polynomial works best with the option value closest to the 'true value'. This result confirms the choice of the fifth-order polynomial model by previous authors such as Bali and Wu (2006). With MC/CIR, we find two optimums, when $\varepsilon \in [-0.5,0.5]$ , the option value is 0.2087 and when $\varepsilon \in [-1.75,1.75]$, the option value is 0.2088, with a 'true' value of 0.2088 computed by the CIR closed-form solution.

To identify the best interval among the two, we simulate equations 1 and 4 one thousand and five thousand times with $\varepsilon \in [-0.5,0.5]$ and with $\varepsilon \in [-1.75,1.75]$ . We obtain Table 3 that suggests that for 5,000 simulations, the smaller the interval, the more accurate will be the option price. Thus, the interval with $\varepsilon \in [-0.5,0.5]$ works best.

Intuitively, we can explain the fact that the narrower the interval of epsilon, the closer the option value to its 'true' value. Roughly speaking, the CIR closed-form solution is obtained by transforming an objective probability measure $P$ in a risk-neutral probability measure $Q$ (Girsanov theorem), by assuming the normality of the distribution of the Wiener process, and by discounting the expected value of the payoff at maturity of the option (Feynman-Kac theorem). Therefore, using MC simulation, the best yield estimates should be the ones found on the estimated yield curve when $\varepsilon \in [0,0]$, e.g. when the diffusion process of equation 1 is equal to zero.

**Table 3. MC simulations with the innovative variance reduction technique when $\varepsilon \in [-0.5,0.5]$ and when $\varepsilon \in [-1.75,1.75]$ for 1,000 and 5,000 simulations. Number of time steps is 500 ($dt$ = 0.01).**

| | CIR closed-form solution | MC /CIR model | MC / Fifth-order polynomial model | Simulation time in seconds |
|---|---|---|---|---|
| 1,000 sim. $\varepsilon \in$[-0.5,0.5] | 0.2088 | 0.2090 | 0.2105 | 115 |
| 1,000 sim. $\varepsilon \in$[1.75,1.75] | 0.2088 | 0.2094 | 0.2105 | 116 |
| 1,000 sim. No bounded interval for $\varepsilon$ (Plain MC) | 0.2088 | 0.2099 | 0.2077 | 117 |
| 5,000 sim. $\varepsilon \in$[-0.5,0.5] | 0.2088 | 0.2090 | 0.2104 | 573 |
| 5,000 sim. $\varepsilon \in$[1.75,1.75] | 0.2088 | 0.2093 | 0.2106 | 571 |
| 5,000 sim. No bounded interval for $\varepsilon$ (Plain MC) | 0.2088 | 0.2096 | 0.2083 | 635 |

SOURCE: ROSTAN AND ROSTAN

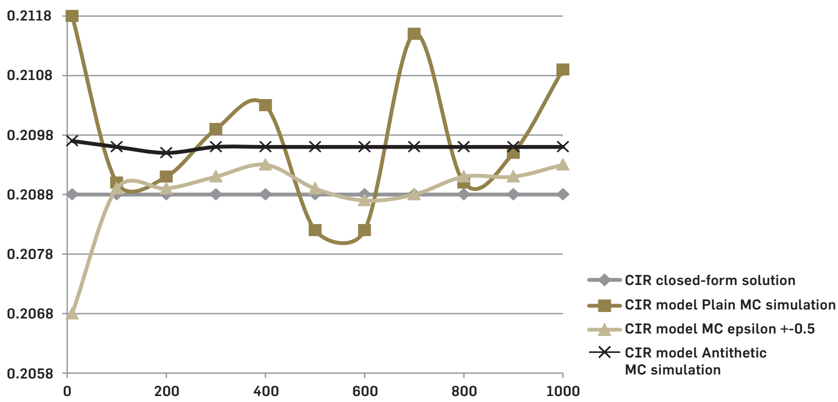
In addition, Table 3 validates the fact that the option price improvement is marginal when we move from 1,000 to 5,000 simulations. There is no improvement at all with MC/CIR model and $\varepsilon \in [-0.5,0.5]$ (0.2090 versus 0.2090), it is slightly better from 0.2105 to 0.2104 with MC/fifth-order polynomial model and $\varepsilon \in [-0.5,0.5]$. However, the simulation time with 5,000 simulations is five times the one with 1,000 simulations. The trade-off between "price accuracy" and "time consuming" means that there is no gain at increasing simulations beyond 1,000. Finally, we note that the simulation time with plain MC simulations is even longer than with our variance reduction technique (635 seconds against 573 seconds). Our technique has the twofold advantage of improving the option price accuracy (0.2090 versus 0.2096 with the plain MC simulation, for a 'true' price of 0.2088) and of being faster than the plain MC simulation.

Summarizing, since MC simulation requires many trajectories in order for the solution to converge towards the 'true' value, our variance reduction technique would therefore be optimal when $\varepsilon \in [-0.5,0.5]$. This result is valid for the pricing of a plain vanilla zero coupon bond option but it may be questionable when we price exotic options such as path-dependent options, with a more skewed pay-offs distribution.

**4.2 Comparing our variance reduction technique to the antithetic variable technique**

To find how good our variance reduction technique is compared to the antithetic variable technique, we refer to Table 1 that shows results with MC and the antithetic variable and to Figures 6 and 7. For a discussion on how to add an antithetic variable to the MC simulation, please refer to the appendix and consult for example Clewlow and Strickland (1998). In the framework of MC/CIR model, Figure 6 compares our reduction technique to: 1) plain MC; 2) MC with antithetic variable.

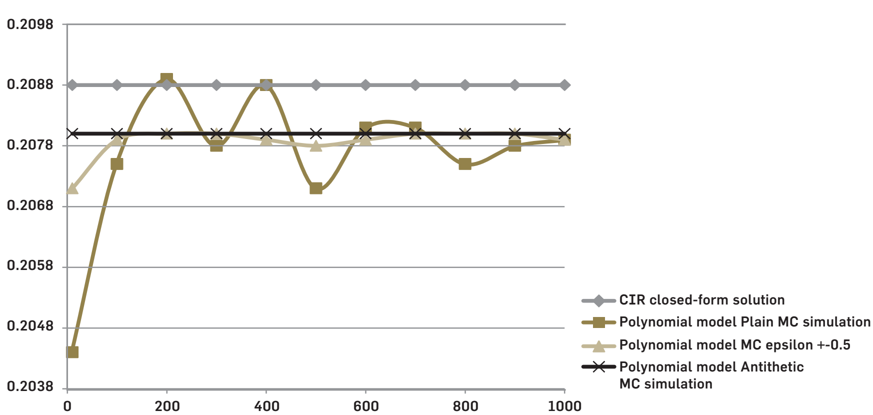
**Figure 6. Using CIR model, comparing our innovative variance reduction technique with $\varepsilon$ [-0.5,0.5] to Plain MC, and MC with antithetic variable, when increasing the number of trajectories from 10 to 1,000. Number of time steps is 500 ($dt$=0.01).**




SOURCE: ROSTAN AND ROSTAN

Figure 6 shows that our technique clearly outperforms Plain MC and MC with antithetic variables. MC with antithetic variable overshoots the CIR closed-form solution; therefore it appears that the antithetic variable skews the results of the Monte Carlo simulation. In the framework of MC/fifth-order Polynomial model, Figure 7 compares our reduction technique to: 1) Plain MC; 2) MC with antithetic variable.

**Figure 7. Using the fifth-order Polynomial model, comparing our innovative variance reduction technique with $\varepsilon$ [-0.5,0.5] to Plain MC, and MC with antithetic variable, when increasing the number of trajectories from 10 to 1,000. Number of time steps is 500 ($dt$ = 0.01).**




SOURCE: ROSTAN AND ROSTAN

Figure 7 shows that the stochastic fifth-order Polynomial model is converging towards an option stead-state value of 0.2080 regardless of the approach (plain, our variance reduction technique or antithetic variable), well below the CIR closed-form solution of 0.2088, which was assumed to be the 'true' value of the bond option in our paper. This assumption of 'true value' can be naturally challenged. If we assume a 'true' value of 0.2080, the fifth-order polynomial model coupled with an antithetic variable works best and beats our innovative reduction technique. A study with real market data could help us identifying the 'true' option value and which model works best. However, this approach suffers from two drawbacks:

**1)** Zero coupon bond options are rarely traded on the exchanges, i.e. most of them are traded on the OTC market where data are more opaque and more difficult to obtain;
**2)** When available on the exchanges, zero coupon bond options trading volume is generally low, i.e. the quotations are not reflecting market fundamental values (i.e. true values).

## 5. Conclusion

Monte Carlo simulation is widely used by practitioners for easiness of implementation, especially for pricing exotic options, particularly path-dependent options such as barriers, lookback or Asian. The closed-form solution is not always available for this type of options. In this situation, Monte Carlo simulation becomes helpful. Our paper proposes an innovative variance reduction technique coupled with Monte Carlo simulation that prices accurately plain-vanilla zero coupon bond options. This technique speeds up the convergence of the simulation and provides better results than Monte Carlo simulation using antithetic variable. Our benchmark is the closed-form solution of Cox Ingersoll and Ross (CIR, 1985). Our paper shows that, when pricing bond options with Monte Carlo simulation, we can constrain the Wiener process inside upper and lower bands to speed up the convergence towards the 'true' option value (the CIR analytical solution). Furthermore, it works best when the bands are drawn at plus or minus 0.5 standard deviations and it is less time consuming than a Plain Monte Carlo simulation.

We introduce an original stochastic fifth-order polynomial model as an improvement to the CIR solution. We show that the option value obtained with this model is generally smaller than the CIR analytical solution. We would like to challenge the result obtained by the CIR closed-form solution but we do not have much alternative to find the 'true' value. Moreover, although our variance reduction technique helps the fifth-order polynomial model to converge faster, it ranks behind the antithetic variable technique in terms of efficiency.

The main contribution of our paper is to provide market practitioners with an efficient variance reduction technique, easy to implement in the context of option pricing. The major challenge of our technique would be to price options in times of high market volatility, when option price needs badly to reflect rare events located in the tails of the distribution. However, we can argue that pricing options in times of volatile markets is a challenge for every option pricing model. Furthermore, our technique is inadequate in risk management, for example to compute Value at Risk with MC simulation, due to the importance of rare events that cannot be discarded.

Further works on the topic could include the comparison of our technique with other variance reduction techniques such as control variate, conditioning, stratified sampling, importance sampling, splitting, quasi-MC, the integration of our innovation technique to the Least-Squares Monte Carlo Method of Longstaff and Schwartz (2001), the extension of our technique to the pricing of other classes of options such as equity, index and currency options, plain-vanilla and exotic, and of other classes of derivative products such as swaps. Testing the Student's t distribution of the innovation term coupled with our variance reduction technique may also be appropriate.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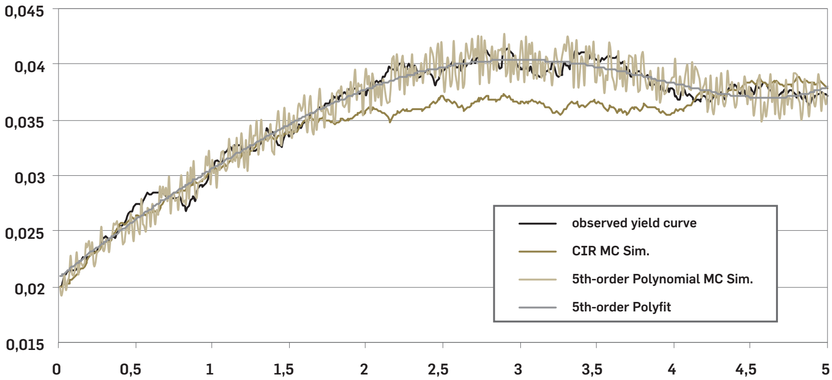
## Appendices

**Figure 8. Simulating two yield curves with MC simulation over 5 years: in brawn using equation 1 (CIR), in beige using equation 4 (Polynomial). In black, the initial simulated yield curve. $\varepsilon \in [-1,1]$**




SOURCE: ROSTAN AND ROSTAN

**A.** Matlab code sample of Monte Carlo simulation using the CIR equation 1 and the antithetic variable:

```
%coding the yield curve with antithetic variable:
zdata(i)= x(1)*(x(2)-y(i))*dt+y(i)^0.5*x(3)*dt ^.5*epsilon;
zdata2(i)= x(1)*(x(2)-y(i))*dt+y(i)^0.5* x(3)*dt ^.5*-epsilon;
%coding the option value; optionvalue1 is computed from zdata;
%optionvalue2 is computed from zdata2
call1=(0.5*optionvalue1+0.5*optionvalue2)/t1;
```

**B.** Matlab code sample of Monte Carlo simulation using the stochastic fifth-order polynomial equation 4 and the antithetic variable:

```
%coding the yield curve with antithetic variable:
k(i+1)=p(1)*w(i)^5+p(2)*w(i)^4+p(3)*w(i)^3+p(4)*w(i)^2+p(5)*w(i)^1+p(6)+v1(i)^0.
5*sigma2*epsilon;
ka(i+1)=p(1)*w(i)^5+p(2)*w(i)^4+p(3)*w(i)^3+p(4)*w(i)^2+p(5)*w(i)^1+p(6)+v1(i)^0.
5*sigma2*-epsilon;
%coding the option value; optionvalue1 is computed from k;
%optionvalue2 is computed from ka
call2=(0.5*optionvalue1+0.5*optionvalue2)/t1;
```